Examination I
MATH 546/701I
26 September 2001

Problem 0

Let m and n be any natural numbers. Prove both of the statements below.

a. $([m,n],m) = m$.

b. $[(m,n),m] = m$.

Problem 1: Core

Let a, b, c, m, and n be integers so that

$$\begin{aligned} a &\equiv c \pmod{m} \text{ and} \\ b &\equiv c \pmod{n}. \end{aligned}$$

Prove that there is an integer d so that

$$\begin{aligned} b &\equiv d \pmod{m} \text{ and} \\ a &\equiv d \pmod{n}. \end{aligned}$$

Problem 2: Core

Let A, B, and C be sets. Let h be a function from A onto B and let g be a function from A onto C such the $\mathrm{KER}\, h = \mathrm{KER}\, g$.

Prove that there is a one-to-one function f from B onto C.

Problem 3

Let n and k be positive integers and h be a homomorphism from $\langle \mathbb{Z}_n, +_n, \cdot_n, -_n, 0, 1\rangle$ onto $\langle \mathbb{Z}_k, +_k, \cdot_k, -_k, 0, 1\rangle$. Prove each of the following statements.

a. $k \leq n$.

b. $h(k) = 0$. [Hint: $h(1 +_n \cdots +_n 1) = h(1) +_k \cdots +_k h(1)$.]

c. $h(a) = r$ for each $a \in \{0, 1, \ldots, n-1\}$, where $a = kq + r$ with $r \in \{0, 1, \ldots, k-1\}$ for some integer q.

d. $k \mid n$.

Extra Credit Problem

Recall that a congruence relation on $\langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$ is an equivalence relation of $\mathbb{Z}$ which respects all the operations listed. Describe all the congruence relations on $\langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$.