FINAL EXAMINATION
MATH 546/701I SECTION 001
15 DECEMBER 2001

PROBLEM 0.

Let m, n, and a be positive integers. Suppose that

$$(m,n) = 1 \qquad \text{and}$$
$$m \mid a \qquad \text{and}$$
$$n \mid a.$$

Prove that $mn \mid a$.

PROBLEM 1: CORE
Let Θ be the set of ordered pairs of integers described by:

$$\langle a, b\rangle \in \Theta \text{ if and only if there is some integer } c \text{ so that } 2 \mid (a-c) \text{ and } 3 \mid (b-c).$$

Prove that Θ is a symmetric relation of the set of all integers.

PROBLEM 2: CORE
Let A, B, and C be sets. Let h be a function from A onto B and let g be a function from A onto C. Let

$$f = \{\langle h(a), g(a)\rangle \mid a \in A\}.$$

Suppose that f is a function from B into C. **Prove** $\text{KER}\, h \subseteq \text{KER}\, g$.

PROBLEM 3.

Let k be an integer with $k > 1$ and let h be a homomorphism from $\langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$ onto $\langle \mathbb{Z}_k, +_k, \cdot_k, -_k, 0, 1\rangle$. Prove that for any integers m and n, if $h(m) = h(n) = 0$, then $(m,n) \neq 1$.

PROBLEM 4: CORE
Let $\mathbf{G}$ be a group, let H be a subgroup of $\mathbf{G}$, and let $g \in G$. Define $K = \{g^{-1}hg \mid h \in H\}$. **Prove** that K is also a subgroup of $\mathbf{G}$.

PROBLEM 5.
Let $\mathbb{R}$ be the set of all real numbers. Let $\mathbb{F}$ be the set of all functions from $\mathbb{R}$ into $\mathbb{R}$. Define addition, negation, and the zero function $\mathbf{0}$ as follows:

$$(f \boldsymbol{+} g)(r) = f(r) + g(r)$$
$$(\boldsymbol{-}f)(r) = -f(r)$$
$$\mathbf{0}(r) = 0$$

for all $f, g \in \mathbb{F}$ and all real numbers r. With these operations, all familiar from calculus classes, $\langle F, \boldsymbol{+}, \boldsymbol{-}, \mathbf{0}\rangle$ becomes a group. (You are not asked to prove that here.)

Define $\Phi : \mathbb{F} \to \mathbb{R}$ via

$$\Phi(f) = f(\pi)$$

for all $f \in \mathbb{F}$. Prove that Φ is a homomorphism from $\langle \mathbb{F}, \boldsymbol{+}, \boldsymbol{-}, \mathbf{0}, \rangle$ into $\langle \mathbb{R}, +, -, 0\rangle$.

PROBLEM 6.
Determine in each part below whether the given permutation is even or odd. Please explain your reasoning.

a. $(1,2)(3,2,1)$
b. $(1,2,3)(1,2,3,4,5)(4,5)$
c. $(2,5)(2,3)(2,4)(2,1)(2,5)$

PROBLEM 7: CORE
Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ be groups. Let h be a homomorphism from $\mathbf{B}$ onto $\mathbf{A}$ and let g be a homomorphism from $\mathbf{C}$ onto $\mathbf{A}$.

Prove that there is an isomorphism α from $\mathbf{B}/\operatorname{KER} h$ onto $\mathbf{C}/\operatorname{KER} g$.
[Warning! Check the directions of the maps g and h.]

Problem 8.
Let $\mathbf{F}$ be a field and let $f(x), g(x)$, and $h(x)$ be polynomials with coefficients from $\mathbf{F}$. Prove that if $(f(x), g(x)) = 1$ and $f(x) \mid h(x)$ and $g(x) \mid h(x)$, then $f(x)g(x) \mid h(x)$.

Problem 9.
Let $\mathbf{F}$ be a field and let $a(x), b(x), c(x) \in \mathbf{F}[x]$. Suppose that φ and ψ are homomorphisms from $\mathbf{F}[x]$ into $\mathbf{F}$ such that

$$\varphi(a(x)) = \varphi(c(x)) \qquad \text{and}$$
$$\psi(b(x)) = \psi(c(x))$$

Find a polynomial $d(x) \in \mathbf{F}[x]$ so that

$$\psi(a(x)) = \psi(d(x)) \qquad \text{and}$$
$$\varphi(b(x)) = \varphi(d(x))$$

Problem 10.
Let $\mathbf{F}$ be a field. Let $\mathbb{F}$ denote the set of all functions from F into F. Define Addition, multiplication, negation, and the constant functions as follows:

$$(f + g)(r) = f(r) + g(r)$$
$$(f \bullet g)(r) = f(r) \cdot g(r)$$
$$(-f)(r) = -f(r)$$
$$\mathbf{0}(r) = 0$$
$$\mathbf{1}(r) = 1$$

for all $f, g \in \mathbb{F}$ and all $r \in F$. With these operations $\langle \mathbb{F}, +, -, \bullet, \mathbf{0}, \mathbf{1} \rangle$ becomes a ring. (You are not asked to prove that here.)

Define $\Phi : \mathbf{F}[x] \to \mathbb{F}$ by letting $\Phi(p(x))$ be the function denoted by the polynomial $p(x) \in \mathbf{F}[x]$. So if $\Phi(p(x)) = \mathbf{p}$, then $\mathbf{p}(r)$ is the result of evaluating $p(x)$ at r, for every $r \in F$ and every $p(x) \in \mathbf{F}[x]$.

Prove that Φ is a homomorphism. Prove this homomorphism is one-to-one if F is infinite.

Problem 11.
In each part below determine whether the given polynomial is an iireducible member of $\mathbb{Q}[x]$.

a. $x^4 - 4x^2 + 2$.
b. $x^4 - 4x^2 + 4$.
c. $x^4 + 5$.

Extra Credit
Let $\mathbf{G}$ be a group and let H, K, and N be subgroups of $\mathbf{G}$ fulfilling all the following conditions:

- $H \subseteq K$,
- $HN = KN$,
- $H \cap N = K \cap N$.

Under these stipulations, prove that $H = K$.