Solutions to High School Math Contest

University of South Carolina, December 8, 2007

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1. **(d)** If x is the number of ladybugs that Jack has and y is the number of ladybugs that Jill has, then $x - y = 5$ and $2x + 7y = 100$. Multiplying the first of these equations by 5 and the second by 2 and adding gives $9(x + y) = 225$. Hence, $x + y = 25$.

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2. **(a)** The sum of Jerry's first 8 scores is $8 \cdot 85$ and the sum of his first 9 scores is $9 \cdot 81$, so he received $9 \cdot 81 - 8 \cdot 85$ on his 9th quiz. You can do the arithmetic or note that the answer ends in a 9.

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3. **(d)** The line with slope 2 passing through $(40, 30)$ has x-intercept $40 - (30/2)$. The line with slope 6 passing through $(40, 30)$ has x-intercept $40 - (30/6)$. So the difference in the x-intercepts is $(30/2) - (30/6) = 10$.

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4. **(e)** If x is the number in the corner square, then the sum of all the numbers in the squares is equal to the sum of the numbers in the five squares aligned vertically plus the sum of the numbers in the five squares aligned horizontally minus x. Hence,

$$1 + 2 + \cdots + 9 = 32 + 20 - x.$$

The sum on the left is 45, so $x = 52 - 45 = 7$.

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5. **(c)** The sum is a little more than $(2/3) + (8/6) + (2/4) = 2.5$.

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6. **(e)** Recall the tests for divisibility by 9 and by 11. Since 342 is divisible by 9, the number $100900b02$ is as well. So b has to be 6. But then $100900b02$ is divisible by 11. Since 342 is not divisible by 11, we deduce $29a031$ must be divisible by 11. So a has to be 5, and $a + b = 11$.

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7. **(c)** Let m be the number of males at Dave's high school, f the number of females, m_s the number of male seniors and f_s the number of female seniors. Then we are given that

$$\frac{1}{10}(m + f) = m, \quad \frac{1}{3}(m_s + f_s) = m_s, \quad p = \frac{m_s}{m} \quad \text{and} \quad q = \frac{f_s}{f}.$$

Dividing by m in the first of these equations and by m_s in the second, we obtain $f/m = 9$ and $f_s/m_s = 2$. Hence,

$$\frac{p}{q} = \frac{m_s f}{m f_s} = \frac{(f/m)}{(f_s/m_s)} = \frac{9}{2}.$$

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8. **(c)** The plane parallel to the base passing through the center of the ellipse will cross the ellipse along its minor axis. If we cut the solid along this plane and flip the top piece that is sliced off 180 degrees around the minor axis, the solid becomes a cylinder with the base of area 9π and height $(2 + 6)/2 = 4$. The volume is therefore 36π.

9. **(e)** The answer follows from $f(g(x)) = g(x)^{g(x)} = (x^{2x})^{g(x)} = x^{2xg(x)} = x^{2x\cdot x^{2x}} = x^{2x^{2x+1}}$.

10. **(b)** If A is the amount of alcohol present in the mixture at some time and the king drinks a proportion p of the mixture, then the king is drinking a proportion p of each of the mixture's parts and, in particular, of the alcohol. Hence, the amount of alcohol that the king leaves in the cup is $A - Ap = A(1-p)$. It follows that the proportion of alcohol left in the cup at the end of the problem is
$$\left(\frac{1}{5}\left(1-\frac{1}{4}\right)+\frac{1}{4}\right)\left(1-\frac{1}{3}\right)+\frac{1}{3} = \frac{3}{5} = 0.60.$$

11. **(d)** Call the expression B. Multiplying B by 2^4 and using that $2\sin\theta\cos\theta = \sin(2\theta)$, we deduce $2^4B = \sin(\pi/2)$. So $B = 1/16$.

12. **(e)** If s is the length of a side of $\triangle ABC$, then its height is $s\sqrt{3}/2$. As the radius of the circle is $s/2$, we deduce $s\sqrt{3}/2 + s/2 = 1$. Hence, $s = 2/(1+\sqrt{3}) = \sqrt{3}-1$.

13. **(b)** If k is a positive integer, then $x^k - y^k$ is divisible by $x - y$. If k is odd, then $x^k + y^k$ is divisible by $x + y$. We use the second of these first to deduce that if n is odd, then $1^n + 4^n$ and $2^n + 3^n$ are both divisible by 5 so that P_n is. Also, if $n = 2m$ where m is odd, then $1^n + 3^n = 1^m + 9^m$ and $2^n + 4^n = 4^m + 16^m$ are both divisible by 5 so that P_n is. Finally, if $n = 4m$ for any positive integer m, then the numbers $1^n - 1$, $2^n - 1 = 16^m - 1$, $3^n - 1 = 81^m - 1$ and $4^n - 1 = 256^m - 1$ are each divisible by 5 so that $P_n - 4$ is and, hence, P_n is not. In other words, P_n is divisible by 5 if and only if n is not divisible by 4. The answer follows.

14. **(b)** One can observe that the numbers 1, 2, 3 and 4 have the stated properties, so the answer "should" be $(1\cdot 4)/(2\cdot 3) = 2/3$. To see that this is indeed the correct answer, we argue as follows. First, observe that if t is non-zero, then $a' = a/t$, $b' = b/t$, $c' = c/t$ and $d' = d/t$ are such that a', b', c' and d' are the first four terms of an arithmetic sequence, a', b' and d' are the first three terms in a geometric sequence, and $a'd'/(b'c') = ad/(bc)$. Take $t = a$ so $a' = 1$. Then a', b', c' and d' being in arithmetic progression implies $c' = 2b' - 1$ and $d' = 3b' - 2$. Since a', b' and d' are in geometric progression, $d' = (b')^2$. Thus, b' satisfies the equation $x^2 = 3x - 2$ or $x^2 - 3x + 2 = 0$. The roots of this quadratic are 1 and 2. Since $b' > a'$, we deduce $b' = 2$ and so $c' = 3$ and $d' = 4$, giving the answer $2/3$. Note that we have actually shown that $\{a, b, c, d\} = \{k, 2k, 3k, 4k\}$ for some $k > 0$.

15. **(b)** Let C be the center of the sphere. The plane passing through A, B and C cuts the sphere in a circle that includes an arc that is the shortest path from A to B. The triangle $\triangle ACB$ is an isosceles triangle with $AC = BC = 12$ and $AB = 12\sqrt{3}$. It is easy to deduce then that $\angle ACB$ has measure $2\pi/3$ radians. The length of the shortest path is the length of the arc, which is $12 \cdot 2\pi/3 = 8\pi$.

16. **(c)** We convert the problem to a simpler one. Let $u = x - 2$ and $v = 3y + 1$. Then the given equation is the same as $u^2 + v^2 = 1$. Also, $4u - 3v = 4x - 9y - 11$. It follows that the maximum

value of $4x - 9y$ where x and y satisfy the equation in the problem is the same as 11 plus the maximum value of $4u - 3v$ where u and v are points on the circle $u^2 + v^2 = 1$. Set $t = v - (4/3)u$ so that $4u - 3v = -3t$. Thus, we want to minimize t. Observe that t is the y-intercept of the line $y = (4/3)x + t$ (where we are abusing notation by using x and y as variables here when they were already defined as numbers in the problem). Thus, we are interested in finding a point (u, v) on the circle $x^2 + y^2 = 1$ that also lies on the line $y = (4/3)x + t$ and we want t minimal. This minimal t is achieved by considering the line $y = (4/3)x + t$ tangent to the circle on the bottom half of the circle. Note that $y = (-3/4)x$ is a line perpendicular to $y = (4/3)x + t$ and passing through the point of tangency of $x^2 + y^2 = 1$ and $y = (4/3)x + t$. We deduce that the minimal t occurs when $v = (-3/4)u$. Setting $v = (-3/4)u$ in the equation $u^2 + v^2 = 1$, we obtain $u = \pm 4/5$. The choice $u = 4/5$ and, hence, $v = -3/5$ corresponds to the tangent point on the bottom part of the circle. Given the above, the answer is $4u - 3v + 11 = 4(4/5) - 3(-3/5) + 11 = 16$.

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17. **(c)** We describe 2 solutions. Let $S = \{1, 2, \ldots, 9\}$. Observe that the sum of the elements of S is divisible by 3. So the problem is the same as asking for the number of ways that 2 elements of S can be chosen so that their sum is divisible by 3 (the other 7 elements of S correspond to a 7-element set as in the problem). Choosing 2 elements of S with sum divisible by 3 corresponds to either choosing 2 numbers divisible by 3 (which can be done in 3 ways) or choosing 1 number that is one more than a multiple of 3 and 1 number that is one less than a multiple of 3 (which can be done in $3 \cdot 3 = 9$ ways). Hence, the answer is $3 + 9 = 12$.

    Alternatively, let S be as before and let T be a 7-element set as in the problem. We say (for the purposes of this problem) that we *adjust* T if we replace each element $t < 9$ in T with $t + 1$ and replace 9 if it is in T with 1. Observe that when we adjust T, each element $t \in T$ is replaced by a number that is congruent to $t + 1$ modulo 3. Also, $7 \equiv 1 \pmod{3}$. If the sum of the elements of T is divisible by 3, then when we adjust T, the new sum of its elements will be 1 modulo 3. If we adjust the new set again, the sum of the elements becomes 2 modulo 3. Imagine now repeating the process of adjusting the elements of T and summing the elements of the set obtained 9 times. On the 9th time, the resulting set will be the original set T that we started with. Including T itself, we will have obtained 9 different 7-element sets, exactly 3 of which have the sum of their elements divisible by 3 (that there are 9 *different* sets requires a little justification). Every 7-element set can be seen to occur once as we vary T over sets as in the problem, and we deduce that exactly $1/3$ of the 7-element subsets of S are such that the sum of their elements is divisible by 3. This answer is therefore

$$\frac{1}{3}\binom{9}{7} = \frac{1}{3} \cdot \frac{9 \cdot 8}{2} = 12.$$

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18. **(c)** Observe that $(0, 0, 0)$ is a triple as in the problem. Now, if one of a, b and c is 0, then the given equations imply each of them is 0. So we suppose next that each is non-zero. The equations $ab = c$ and $ac = b$ imply $a^2bc = bc$ so that $a^2 = 1$. Hence, $a \in \{1, -1\}$. Similarly, $b \in \{1, -1\}$ and $c \in \{1, -1\}$. One checks that these imply that either $(a, b, c) = (1, 1, 1)$ or exactly one of a, b and c is 1 and the other two are -1. These observations lead to the total number of tuples being 5.

19. **(d)** The equation $(x^2+y^2)(x^3+y^3)=12$ is equivalent to

$$\big((x+y)^2-2xy\big)(x+y)\big((x+y)^2-3xy\big)=12.$$

Letting $u=xy$ and recalling $x+y=1$, we deduce $(1-2u)(1-3u)=12$. Hence, $6u^2-5u-11=0$. The quadratic factors to give $u=11/6$ or $u=-1$. As x and y are real roots of the quadratic

$$(t-x)(t-y)=t^2-(x+y)t+xy=t^2-t+u,$$

we must have that its discriminant $1-4u$ is ≥ 0. Hence, $u=-1$ and $x^2+y^2=(x+y)^2-2u=1+2=3$.

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20. **(c)** Since $\triangle PAB$ is isosceles, $\angle PAB=\angle PBA$. Given $\overleftrightarrow{AP}$ bisects $\angle CAB$ and $\overleftrightarrow{BP}$ bisects $\angle CBA$, we deduce $\angle CAB=\angle CBA$. Thus, $\triangle CAB$ is isosceles. Let T be a point on segment $\overline{AB}$ such that $\overline{CT}$ is an altitude for $\triangle CAB$. Note that necessarily P is on $\overline{CT}$. Set $\theta=\angle PBA$ and $u=\sin\theta$. Then $\angle PBC=\theta$ and $\angle PCB=\pi/2-2\theta$. Using $\triangle PBC$ and the Law of Sines, we deduce

$$\frac{\sin\theta}{3}=\frac{\sin(\pi/2-2\theta)}{\sqrt{3}} \implies \sin\theta=\sqrt{3}\,\cos(2\theta) \implies u=\sqrt{3}(1-2u^2).$$

This last equation can be rewritten as $(\sqrt{3}u-1)(2u+\sqrt{3})=0$. Since $u>0$, we obtain $u=1/\sqrt{3}$. This implies $PT=1$ and, consequently, $TB=\sqrt{2}$. Hence, the area of $\triangle ABC$ is $CT\cdot TB=4\sqrt{2}$.

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21. **(b)** The minimal path is a line segment from $(2,5)$ to a point, say P, on the x-axis together with a line segment from P to a point Q on the given circle. The line $\overleftrightarrow{PQ}$ passes through $(-6,10)$, the center of the circle (which follows from PQ being the shortest path from P to the circle). If $Q=(a,b)$, then we set $Q'=(a,-b)$, the reflection of Q about the x-axis. This reflection takes the segment from P to Q to the segment from P to Q'. Also, $\overleftrightarrow{PQ'}$ passes through $(-6,-10)$. There is in fact a 1-1 correspondence between paths from $(2,5)$ to P to Q with $\overleftrightarrow{PQ}$ passing through $(-6,10)$ and paths from $(2,5)$ to P to Q' with $\overleftrightarrow{PQ'}$ passing through $(-6,-10)$ given by the reflection of the second part of the paths about the x-axis. We deduce that the length of the path from $(2,5)$ to P to Q is the distance from $(2,5)$ to P plus the distance from P to $(-6,-10)$ minus 4 (the radius of the given circle). This length is minimized by taking P to be on the line passing through $(2,5)$ and $(-6,-10)$. As the distance between these two points is $\sqrt{8^2+15^2}=17$, the answer is $17-4=13$.

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22. **(a)** First, we show that one of p and q is 2. If both p and q are odd, then $n^2+1=(p^2+1)(q^2+1)$ is even so that n is also odd. Hence, there is an integer t such that $n=2t+1$ so that $n^2+1=4t^2+4t+2$. But this means n^2+1 is not divisible by 4 whereas $(p^2+1)(q^2+1)$ clearly is. Thus, we must have one of p or q is 2. We now have $n^2+1=5(x^2+1)$ where either $x=p$ or $x=q$. Note that $x\geq 3$. We deduce that $5x^2=n^2-4=(n-2)(n+2)$. Since x is prime and $n+2>n-2$, we obtain that $n-2\in\{1,5,x\}$ and $n+2=5x^2/(n-2)$. Since also $(n+2)-(n-2)=4$, we get that one of $5x^2-1$, x^2-5 and $4x$ equals 4. As x is prime, we obtain $x=3$. Thus, $(p,q)=(2,3)$ or $(3,2)$. Each pair gives a solution to the equation $n^2+1=(p^2+1)(q^2+1)$, implying the answer.

23. **(b)** Note that $135 = 3^3 \cdot 5$. If p^u divides $n!$ and p^{u+1} does not, then $u = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$, where $\lfloor x \rfloor$ denotes the largest integer $\leq x$. This formula can be established by using that there are exactly $\lfloor n/p^j \rfloor$ multiples of p^j that are $\leq n$. Let r and s be the positive integers for which 3^r and 5^s divide 2007! but 3^{r+1} and 5^{s+1} do not divide 2007!. Then

$$r = \left\lfloor \frac{2007}{3} \right\rfloor + \left\lfloor \frac{2007}{3^2} \right\rfloor + \cdots = 669 + 223 + 74 + 24 + 8 + 2 = 1000.$$

Similarly, $s = 500$. If $135^k = 3^{3k} \cdot 5^k$ divides 2007! and $3^{3k+3} \cdot 5^{k+1}$ does not, then $k = 333$.

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24. **(a)** The conditions imply $a_1 + a_2 + \cdots + a_r = 230$ and $a_1 a_2 \cdots a_r = 2007$. One checks that $2007 = 3^2 \cdot 223$, where 223 is prime. We deduce that some a_j is 223, two a_j's are 3, and the remaining a_j's are 1. Since their sum is $230 = 223 + 3 + 3 + 1$, we obtain $r = 4$.

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25. **(a)** Label the starting square A and the shaded square D. Let B be the square that is down 2 squares from A and to the right 3 squares from A (this is the top left square of the bottom 4 by 4 grid). Let C be the square that is down 3 squares from A and to the right 2 squares from A. Then each sequence of 11 moves from A to D passes through exactly one of B or C. The number of paths from A to B to D is $\binom{5}{2}$ (it takes exactly 5 moves to go from A to B with exactly 2 of the moves being in a vertical direction) times $\binom{6}{3}$ (it takes exactly 6 moves to go from B to D with exactly 3 of the moves being in a horizontal direction). Note that a sequence of moves (to the right or downward) from C to D necessarily begins with a move to the square adjoining C on the right. We deduce in a similar way that the number of paths from A to C to D is $\binom{5}{2}$ times $\binom{5}{2}$. Hence, the answer is

$$\binom{5}{2} \cdot \binom{6}{3} + \binom{5}{2} \cdot \binom{5}{2} = 10 \cdot 20 + 10 \cdot 10 = 300.$$

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26. **(d)** Let n be as large as possible so that every 100-element subset of S contains two integers differing by 25. Observe that the set

$$A = \{1, 2, \ldots, 25\} \cup \{51, 52, \ldots, 75\} \cup \{101, 102, \ldots, 125\} \cup \{151, 152, \ldots, 175\}$$

has 100 elements no two of which differ by 25. So $n \leq 174$. We show now that $S = \{1, 2, \ldots, 174\}$ has the property that every 100-element subset of S contains two integers differing by 25. For $0 \leq j \leq 24$, let T_j be the subset of S consisting of the integers from S that have a remainder of j when we divide by 25. The sets T_j have no common elements. If S' is a subset of S consisting of exactly 100 elements, then there are exactly 74 elements of S not in S'. These must lie in the 25 sets T_j. One of the sets T_j contains at most 2 of these 74 elements of S not in S' (by the pigeon-hole principle). Fix such a j. Since T_j contains at least 6 elements, the intersection $T_j \cap S'$ contains two consecutive elements of T_j, that is two elements of T_j that differ by 25. Thus, S has the property that every 100-element subset of S contains two integers differing by 25.

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27. **(d)** Observe that

$$\frac{n}{(n-2)! + (n-1)! + n!} = \frac{1}{(n-2)!} \times \frac{n}{1 + (n-1) + n(n-1)}$$

$$= \frac{1}{(n-2)!} \times \frac{1}{n} = \frac{1}{(n-1)!} \times \left(1 - \frac{1}{n}\right) = \frac{1}{(n-1)!} - \frac{1}{n!}.$$

Hence, the sum in the problem is the telescoping series

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + \frac{1}{2006!} - \frac{1}{2007!} = \frac{1}{2!} - \frac{1}{2007!},$$

which can be rewritten as in choice (d).

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28. **(e)** Let r be the number of red marbles and b the number of blue marbles. The number of ways of choosing two marbles is $\binom{r+b}{2}$. The number of ways of choosing two different color marbles is rb. Hence, we have

$$\frac{1}{2} = \frac{rb}{\binom{r+b}{2}} = \frac{2rb}{(r+b)(r+b-1)}.$$

This can be rewritten as

$$b^2 - (2r+1)b + r^2 - r = 0.$$

Since the number of blue marbles must be an integer that is a root of the above equation, we deduce that the discriminant of this quadratic is a square. In other words, there is an integer m such that

$$(2r+1)^2 - 4(r^r - r) = m^2 \qquad \Longrightarrow \qquad 8r + 1 = m^2.$$

The only $r \in \{7, 9, 11, 13, 15\}$ for which this equation holds for some integer m is 15, so the answer is 15. One can check that if $r = 15$ and $b = 10$, the probability that two randomly chosen marbles have different colors is in fact 0.5.

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29. **(e)** Substituting $x = 5$ into the given equation, we deduce $7 \cdot 10 \cdot 12 \cdot u(5) = m$. Substituting $x = 7$ into the given equation, we deduce $9 \cdot 12 \cdot 14 \cdot u(7) = m$. It follows that m must be divisible by each of $7 \cdot 10 \cdot 12$ and $9 \cdot 12 \cdot 14$ which implies that $2^3 \cdot 3^3 \cdot 5 \cdot 7 = 7560$ divides m. Of the choices, we see that only (e) is possible here. Verifying that such a $u(x)$ and $v(x)$ exist requires a little more work, but we note that one can take $u(x) = x^2 - 14x + 54$ and $v(x) = x^2 + 14x + 54$.

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30. **(d)** We consider the size of m relative to n. If $m = n$, then n is a good number if and only if $7n = 133$. In this case, we get 19 is a good number. Now, suppose that $m \geq n + 1$. Then

$$n^3 + 7n - 133 = m^3 \geq (n+1)^3 = n^3 + 3n^2 + 3n + 1$$

so that $3n^2 - 4n + 134 \leq 0$. For positive integers n, it is easy to see that this is impossible. It remains to consider $m \leq n - 1$. For such m, we have

$$n^3 + 7n - 133 = m^3 \leq (n-1)^3 = n^3 - 3n^2 + 3n - 1$$

which implies $3n^2 + 4n - 132 \leq 0$. It follows here that $n \leq 6$. One checks that

$$6^3 + 7 \cdot 6 - 133 = 5^3 \quad \text{and} \quad 5^3 + 7 \cdot 5 - 133 = 3^3,$$

so 6 and 5 are good numbers. For $n \leq 4$, we see that $n^3 + 7n - 133 < 0$ and, hence, $n^3 + 7n - 133$ cannot equal m^3 for a positive integer m. Therefore, the sum of all good numbers is $19 + 6 + 5 = 30$.