Solutions to USC's 21st High School Math Contest

1. (d) Just keep track of the last two digits of the consecutive powers of 5 and note that $5^2 = 25$ and $5 \times 25 = 125$.

2. (d) Suppose the vertices of the rectangle are labeled A, B, C, and D in such a way that AB is the shorter side. Let O be the intersection point of the two diagonals. Then $\triangle AOB$ is equilateral.

3. (e) Consider the right triangle $\triangle ABC$. By the Pythagorean Theorem $BC = 3$. Then, $BF = 6$. Considering the right triangle $\triangle ABF$ we get $AF = 2\sqrt{13}$.

4. (b) $x = -y \neq 0$, so $\dfrac{x}{y} = -1$. Also, $(-1)^{2007} = -1$.

5. (a) Label the 4 small rectangles R_1, R_2, R_3, and R_4 so that the areas of R_1, R_2, and R_3 are 8, 6, and 9 respectively and R_4 is the rectangle whose area we want to find. Denote the area of R_4 by x. Then $8 \cdot 9 = 6 \cdot x$. (Note that R_1 and R_2 have the same base and so do R_3 and R_4. Also, R_1 and R_4 have the same height and so do R_2 and R_3.)

6. (e) Joe had \$100 in his left and \$100 in his right pocket at the end. So, he had \$120 in his left pocket (and \$80 in his right pocket) before moving \$20 from his left to his right pocket. Thus, he had \$160 in his left pocket originally (solve $x - x/4 = 120$).

7. (b) Note that the surface area of the shape is the same as the surface area of the large cube. Also, the large cube has side 5, and surface area $6 \times 5^2 = 150$.

8. (a) Let S_1 be the number of students who play exactly one of the three sports; S_2 - the number of students who play exactly two of the three sports; and let S_3 be the number of students who play all three sports. We know that $S_3 = 15$ and $S_1 + S_2 + S_3 = 100$ (the total number of students). Also in the sum $50+45+50$ (soccer players + basketball players + volleyball players), each student who plays exactly one sport is counted once, the ones who play exactly two sports are counted twice, and those who play all three sports are counted three times. Thus, $50 + 45 + 50 = S_1 + 2S_2 + 3S_3$. Note that $S_2 = (S_1 + 2S_2 + 3S_3) - (S_1 + S_2 + S_3) - 2S_3 = 15$.

9. (c) Let $AB = x$. Since $ABCD$ is a square, all 4 right triangles $\triangle PBQ$, $\triangle QCR$, $\triangle RDS$, and $\triangle SAP$, have base x and height $2x$. Thus, all 4 triangles have area x^2, and so does the square $ABCD$. Hence, the square $PQRS$ has area $5x^2$.

10. (a) Suppose we call what's left from a grape after removing all water "waterless content". Then 34 pounds of fresh grapes contain 6.8 pounds "waterless content". Also, x pounds dried grapes contain $x - 0.15x = 0.85x$ pounds "waterless" content". Solving the equation $0.85x = 6.8$ we get $x = 8$.

11. (b) The first inequality holds when $\sin\theta > 0$ and the second - when $\cos\theta < 0$. If θ is an angle between $0°$ and $360°$, then $\sin\theta$ is positive when $0° < \theta < 180°$; and $\cos\theta$ is negative when $90° < \theta < 270°$.

12. (c) The middle number is 140 and the seven numbers are $137, 138, 139, \cdots, 143$. Now $138, 140$, and 142 are even, 141 is divisible by 3, and 143 - by 11. The remaining numbers 137 and 139 are prime (it is easy to check that they are not divisible by 2,3,5,7 and 11).

13. (c) Let M be the number of days it rained in the morning; A - the number of days it rained in the afternoon; and let N be the number of days when it did not rain. We have $M + A = 7$ (days when it rained); $M + N = 5$ (afternoons when it did not rain); and $A + N = 6$ (mornings when it did not rain). Adding the equations we get $2(M + A + N) = 18$, $M + A + N = 9$.

14. (b) Substituting in the formula we get $b_2 = -3$, then $b_3 = -1/2$, $b_4 = 1/3$, and $b_5 = 2 = b_1$. Thus, the sequence is periodic with a period 4, and $b_1 = b_5 = \cdots = b_{2005} = 2$, $b_2 = b_6 = \cdots = b_{2006} = -3$.

15. (a) Let S be the area of one of the shaded regions. Then the smallest circle has area $4S$, the middle - area $8S$, and the largest circle - area $12S$. Since the smallest circle has radius 1, its area is π. Thus, $\pi = 4S$, and $S = \pi/4$. So, the middle circle has area 2π, and radius $\sqrt{2}$, and the largest circle - area 3π and radius $\sqrt{3}$.

16. (e) Denote the product by P. Note that $\left(1 - \frac{1}{k^2}\right) = \left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right) = \frac{k-1}{k}\frac{k+1}{k}$. Thus, $P = \frac{1}{2}\frac{3}{2} \cdot \frac{2}{3}\frac{4}{3} \cdots \frac{49}{50}\frac{51}{50}$. Rearranging factors, we get $P = \frac{1}{2}\frac{2}{3} \cdots \frac{49}{50} \cdot \frac{3}{2}\frac{4}{3} \cdots \frac{51}{50} = \frac{1}{50} \cdot \frac{51}{2} = \frac{51}{100}$.

17. (b) Let k be the number of all the people who are both nice and wise. Then, the number of wise people is $4k$ and the number of nice people is $2k$. The number of people who are either wise or nice or both is $4k + 2k - k = 5k$. If the number of all people is x, then $x - 0.25x = 5k$, so $x = 20k/3$.

18. (e) $0 = (x - 1)(x^2 + x + 1) = x^3 - 1$. Thus, $x^3 = 1$, and $(1 + 1)^3 = 8$.

19. (d) The last digit of $11^n - 1$ is 0, so 5 always divides $11^n - 1$. Also, checking the remainders when the powers of 11 are divided by 3 and 7 we get that 3 divides $11^n - 1$ exactly when n is even, and 7 divides $11^n - 1$ exactly when 3 divides n.

20. (c) The sum of the roots of the polynomial is 8. Since the roots are distinct positive integers, then they are either $1, 2, 5$ or $1, 3, 4$. Thus, the polynomial is either $(x - 1)(x - 2)(x - 5)$ or $(x - 1)(x - 3)(x - 4)$.

21. (d) Let W be the total weight of all fish the man caught. The fish taken by the cat had weight $30\%W$. If the cat took 4 fish or more, then their average weight must have been at most $7.5\%W$ which is less than the average weight of the 5 lightest fish. If the cat took 1 or 2 fish, then their average weight must have been at least $15\%W$ which is more than the average weight of the 2 heaviest fish. Thus, the cat took 3 fish.

22. (e) There were 45 games played in the tournament ($\binom{10}{2}$ games). Now, the total number of points awarded in one game is either 3 (if one team wins), or 2 (when the teams tie the game). So, the total number of points received is $3 \times 45-$(the number of games ended in a tie). Thus, exactly 5 games ended in a tie.

23. (a) Denote the radius of the identical circles by r. Then the triangle with vertices the center of the middle circle on the bottom row and the centers of the two circles in the middle row, is an equilateral triangle with side $2r$. Similarly, the triangle with vertices the center of the top circle and the centers of the two circles in the middle row, is an equilateral triangle with side $2r$. Thus, the height of the shape equals $r + h + h + r = 2(r + h)$, where h is the height of a equilateral triangle with side $2r$. It

is easy to show that $h = \sqrt{3}r$. Thus, $r = \frac{2}{2(1+\sqrt{3})} = \frac{1}{1+\sqrt{3}}$.

24. (d) Substitute $x = 2$ to get $f(2) + f(-1) = 2$. Next, substitute $x = -1$ to get $f(-1) + f(1/2) = -1$. Finally, substitute $x = 1/2$ to get $f(1/2) + f(2) = 1/2$. Solving the above equations for $f(2)$ we obtain $f(2) = 7/4$.

25. (d) Jerry wrote down a total of $2^7 - 1 = 127$ numbers (0 is missing). Add 0 to the numbers Jerry wrote. We divide the resulting 128 numbers in pairs. If x is one of the numbers Jerry wrote then we put it in pair with $1111111 - x$ (whose digits are also 0s and 1s). For example 1100101 is paired with 11010. In this way, we divide the numbers into 64 distinct pairs so that each number appears exactly once. Note that there are exactly 7 digits 1 in each pair. So, Jerry wrote the digit 1 exactly $64 \times 7 = 448$ times.

26. (c) Let D be the point on the base of the cube such that CD is perpendicular to the base. Let s be the length of the sides of the cube. Then $AB = BC = AD = \sqrt{2}s/2$. Using the Pythagorean Theorem in $\triangle ADC$, we get $AC = \sqrt{3}s/\sqrt{2}$. By the Law of Cosines $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos(\angle ABC)$. We get $\cos(\angle ABC) = -1/2$, so $\angle ABC = 120°$.

27. (e) We have $12 @ 5 = 5 @ 12 = 4 @ 12 + 13 = 3 @ 12 + 26 = 2 @ 12 + 39 = 1 @ 12 + 52 = 0 @ 12 + 65 = 12 @ 0 + 65 = 12 + 65 = 77$.

28. (b) If $x < 0$ the equation has no solution. So, let $x \geq 0$. If $\sqrt{3+x} > x$, then $3 + \sqrt{3+x} > 3 + x$, and $\sqrt{3+\sqrt{3+x}} > \sqrt{3+x} > x$. Add 3 to the outer parts of the last inequality and take square root to get $\sqrt{3+\sqrt{3+\sqrt{3+x}}} > \sqrt{3+x} > x$. So, the equation has no solution if $\sqrt{3+x} > x$. Similarly, if $\sqrt{3+x} < x$, the equation has no solution. Also, if $\sqrt{3+x} = x$ and $x \geq 0$, then x is a solution. Well, the quadratic equation $3 + x = x^2$ has one positive and one negative solution. The positive solution of $3 + x = x^2$ is the only solution of the equation in problem 28.

29. (c) Consider the polynomial $g(x) = xf(x) - 1$. Each of the numbers $1, 2, \ldots, 2007$ is a root of $g(x)$. Since the degree of $g(x)$ is 2007,

$$g(x) = xf(x) - 1 = c(x-1)(x-2)\cdots(x-2007)$$

for some constant c. Substitute $x = 0$ in the above equation. We obtain $-1 = c \cdot (-1)(-2)\cdots(-2007)$, or $c = 1/2007!$. Now, if we substitute $x = 2008$ we get $g(2008) = 2008f(2008) - 1 = \frac{1}{2007!} \cdot 2007 \cdots 1 = 1$. Thus, $f(2008) = 2/2008$.

30. (b) The answer is 12 rows. If 59 schools send 34 students each, then we can sit at most 5 groups of students in the same row, so we will need 12 rows. Next, 12 rows are sufficient. Assume that this is not the case. Suppose the groups of students are seated like this: first the largest group, then the second largest group, then the third largest group, etc. Suppose we run out of space - there are not enough seats in any row to seat together the next group. Suppose the first group that can not be seated together is the kth group and it consists of N students. Then $k \geq 61$ since any row fits at least 5 groups. Also, $N \leq 2006/k \leq 2006/61 < 33$ (all groups already seated are no smaller than the kth group). So, $N \leq 32$. Since there is not enough space in any of the 12 rows to seat the kth group, then there must be at least 168 students seated in each of the 12 rows. Now, $12 \times 168 = 2016 > 2006$ a contradiction. So, 12 rows are sufficient.