

## SOLUTIONS TO USC'S 2002 HIGH SCHOOL MATH CONTEST

1. **(b)** The answer follows from  $8 \times 8^8 = 8^9$ .
2. **(c)** The conditions of the problem imply  $\frac{m+9}{7-m} = m$ . This simplifies to  $m^2 - 6m + 9 = 0$ . Since  $m^2 - 6m + 9 = (m-3)^2$ , we deduce  $m = 3$ .
3. **(d)** Since  $\triangle BCD$  is an equilateral triangle, the measure of  $\angle BCD$  is  $60^\circ$ . Also,  $\triangle ABC$  is a right triangle. It follows that  $AB = \sqrt{3}$ .
4. **(d)** If  $w$  is the number of women in the class and  $N$  the total number of students, then the number of men in the class is  $N - w$ . The total number of points earned by the class on the test can be computed in two ways from the information given, as  $80N$  and as  $83w + 71(N - w)$ . Hence,  $83w + 71(N - w) = 80N$  so that  $12w = 9N$ . The proportion of students that are women is  $w/N = 9/12 = 75\%$ .
5. **(b)** From  $3^{abcdef} = (3^a)^{bcdef} = 4^{bcdef} = 5^{cdef} = 6^{def} = 7^{ef} = 8^f = 9$ , we deduce  $abcdef = 2$ .
6. **(a)** Just list the arrangements: SIX, SXI, ISX, IXS, XIS, and XSI.
7. **(d)** The number 141 is an obtainable score since a person can get 28 problems correct, leave one blank, and miss the other. A score  $S$  is obtainable if and only if there are nonnegative integers  $x$  (the number correct) and  $y$  (the number blank) satisfying  $5x+y = S$  and  $x+y \leq 30$ . If  $S = 147$ , then  $y$  is at least 2 so that  $5x+y = 5(x+y) - 4y \leq 150 - 4y \leq 142$ , which is impossible. If  $S \in \{139, 143, 144\}$ , then  $y \geq 3$  so that  $5x+y = 5(x+y) - 4y \leq 150 - 4y \leq 138$ , which is impossible. Hence, only Drew can be correct.
8. **(c)** Let  $a$  and  $b$  be the legs of the right triangle. Then the hypotenuse is  $\sqrt{a^2 + b^2}$ . Hence, the sums of the squares of its sides is  $2a^2 + 2b^2 = 578$ . It follows that  $a^2 + b^2 = 289 = 17^2$  so that the hypotenuse is 17. Since the perimeter is 40, we deduce that  $a + b = 23$ . Also,  $2ab = (a+b)^2 - (a^2 + b^2) = 23^2 - 289 = 240$ . Thus,  $ab = 120$ . It follows that  $a$  and  $b$  are the roots of  $x^2 - 23x + 120 = (x-8)(x-15)$ . Hence, the smaller side is 8.
9. **(c)** Let  $x_0$  be such that  $f(x_0) = 2$ . Then  $f(f(x_0)) \cdot (1 + f(x_0)) = -f(x_0)$  is equivalent to  $f(2) \times 3 = -2$ . Hence,  $f(2) = -2/3$ .
10. **(d)** Adding 1 to each expression gives
 
$$\frac{a+b+c}{c} = \frac{a+b+c}{a} = \frac{a+b+c}{b}.$$
 Since we are given that  $a$ ,  $b$ , and  $c$  are distinct, the above holds only if  $a + b + c = 0$ . But then  $a + b = -c$  so that  $(a+b)/c = -1$ . The answer is therefore  $-1$ . (Observe that the quotients in the problem are  $-1$  whenever each of  $a$ ,  $b$ , and  $c$  is non-zero and  $a + b + c = 0$ . For example,  $a = 1$ ,  $b = 2$ , and  $c = -3$  will give that each quotient in the problem is  $-1$ .)
11. **(a)** The conditions in the problem imply  $y^2 - x = x^2 - y$  so that  $(x+y)(x-y) = x^2 - y^2 = -(x-y)$ . Since  $x$  and  $y$  are given as being distinct, we can divide by  $x-y$  to obtain  $x+y = -1$ . The conditions in the problem now imply  $x^2 + y^2 = (2002+y) + (2002+x) = 4004 - 1 = 4003$ . Thus,  $2xy = (x+y)^2 - (x^2 + y^2) = 1 - 4003 = -4002$ . Dividing by 2 gives  $xy = -2001$ . (In fact,  $x$  and  $y$  must be  $(\sqrt{8005} - 1)/2$  and  $(-\sqrt{8005} - 1)/2$  in some order.)
12. **(a)** Observe that  $\alpha^4 + \alpha^2 - 1 = 0$  implies  $\alpha^6 + \alpha^4 - \alpha^2 = 0$  (by multiplying by  $\alpha^2$ ) and  $\alpha^2 = 1 - \alpha^4$  (by adding  $1 - \alpha^4$  to both sides of the original equation). Hence,  $\alpha^6 + \alpha^4 = \alpha^2 = 1 - \alpha^4$  so that  $\alpha^6 + 2\alpha^4 = 1$ .
13. **(e)** Initially, there are an even number of white circles. Changing the color of two white circles to black, changing

the color of two black circles to white, or changing the color of one white circle to black and one black circle to white will each result in the number of white circles remaining even. So the number of white circles will remain even after any sequence of operations as described in the problem. Since there are an odd number of white circles in part (e), this row cannot result from applying the operation any number of times. (The other rows can be checked to result from applying the operation one or more times.)

14. **(a)** For negative values of  $x$ , the left-hand side is positive since it equals  $(x^2 - 3)^2$  and the right-hand side is negative since  $x^3$  and  $x$  will both be negative when  $x$  is. Hence, the two sides cannot be equal for a negative value of  $x$ .
15. **(c)** Since all the coefficients are positive, the polynomial cannot have positive rational roots. The rational root test then implies that the only rational roots must be either  $-2$ ,  $-1$ , or/and  $-1/2$ . The value of the polynomial is odd for  $x = -1$ , so  $-1$  is not a root. One can check that both  $-2$  and  $-1/2$  are roots directly (but once  $-2$  is checked, one can divide the equation by  $2^4$  to see that  $-1/2$  is also a root).
16. **(b)** Let  $h_1$  be the height of  $\triangle DAC$  drawn from  $D$  to  $\overline{AC}$ . Let  $h_2$  be the height of  $\triangle BCA$  drawn from  $B$  to  $\overline{AC}$ . By the areas given,  $h_2 \times OC = 12 = 3 \times 4 = 3 \times h_1 \times OC$ . Thus,  $h_2 = 3h_1$ . Since the area of  $\triangle AOB$  is 3, the area of  $\triangle DOA$  is  $(1/2)h_1 \times OA = (1/3)(1/2)h_2 \times OA = (1/3) \times 3 = 1$ . (Note that the ratio of the area of  $\triangle AOB$  to the area of  $\triangle DOA$  is equal to the ratio of the area of  $\triangle BOC$  to the area of  $\triangle COD$ .)
17. **(d)** One can simplify the equation to obtain  $x = \frac{21x + 13}{13x + 8}$  so that  $13x^2 + 8x = 21x + 13$ . In other words,  $13x^2 - 13x - 13 = 0$  or  $x^2 - x - 1 = 0$ . The sum of the roots of  $x^2 - x - 1$  is minus the coefficient of the second term divided by the coefficient of the first term. In other words, the answer is 1. Alternatively, one can show (by induction) that any expression of the form

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots 1 + \frac{1}{1 + \frac{1}{x}}}}}$$

can be rewritten as  $\frac{(a+b)x+a}{ax+b}$  for some numbers  $a$  and  $b$ , and this is sufficient to get the answer to the problem. Indeed, the answer would be the same if the number of divisions that occur in the expression on the right is changed (as long as at least one division occurs).

18. **(b)** The answer follows from

$$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2 - (a + b)(c + d) - (a + c)(b + d) - (a + d)(b + c) = 10^2 - 16 - 21 - 24 = 39.$$

One particular solution to the equations is  $a = 11/2$ ,  $b = 5/2$ ,  $c = 3/2$ , and  $d = 1/2$ .

19. **(e)** Observe that  $a \cdot [a] = 17$  implies  $4 < a < 5$ . Hence,  $[a] = 4$ . Now,  $a \cdot [a] = 17$  implies  $a = 17/4$ . Similarly,  $b \cdot [b] = 11$  implies  $b = 11/3$ . Therefore,  $a - b = (17/4) - (11/3) = 7/12$ .
20. **(b)** A necessary and sufficient condition for  $ax^2 - 2x + a < 0$  for all real  $x$  is that the graph of  $y = ax^2 - 2x + a$  is below the  $x$ -axis. This is equivalent to having  $a < 0$  (so the graph is of a parabola opening downward) and  $4 - 4a^2 < 0$  (so the polynomial  $ax^2 - 2x + a$  has no real roots) which in turn is equivalent to having  $a < -1$ .
21. **(e)** From  $\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$ , we see that  $x = e^{i\pi/1001}$  and  $x = e^{-i\pi/1001}$  satisfy the given equation. Also,  $e^{i\theta} = \cos \theta + i \sin \theta$  implies that these values of  $x$  are distinct. Since the original equation is equivalent to a quadratic in  $x$ ,  $\alpha$  must be one of these two values. In either case,  $\alpha^{2002} + 1/\alpha^{2002} = e^{i2\pi} + e^{-i2\pi} = 1 + 1 = 2$ .

22. (e) Observe that for an integer  $n \geq 0$ ,  $(n + 1)! = (n + 1) \cdot n! = n \cdot n! + n!$ . It follows that

$$\begin{aligned} 1001! &= 1000 \cdot 1000! + 1000! = 1000 \cdot 1000! + 999 \cdot 999! + 999! \\ &= \dots = 1000 \cdot 1000! + 999 \cdot 999! + \dots + 2 \cdot 2! + 1 \cdot 1! + 1!. \end{aligned}$$

In other words, the sum in the problem is the same as  $1001! - 1$ . Since  $1001!$  is divisible by 2002, the remainder when dividing  $1001! - 1$  by 2002 is 2001. (Alternatively, one can determine the remainder by using that  $2002 = 2 \times 7 \times 11 \times 13$  and considering the remainder when dividing by each of the primes 2, 7, 11, and 13.)

23. (c) Let  $j$  denote John's age when Tammy was 4, and let  $m$  denote Martha's age when Tammy was 4. Let  $a$  be the number of years after Tammy was 4 when Martha was twice as old as Tammy and John was five times as old as Tammy. Let  $b$  be the number of years after Tammy was 4 when John was twice as old as Martha. Then we have the following equations: (i)  $j = 3m$ , (ii)  $m + a = 2(4 + a)$ , (iii)  $j + a = 5(4 + a)$ , and (iv)  $j + b = 2(m + b)$ . We want to know the value of  $4 + b$ . From (iv),  $b = j - 2m$  so that, from (i),  $b = m$ . From (ii),  $m = 8 + a$ . Using (i) in (iii), we also have  $3m = 20 + 4a$ . So  $b = m = 4m - 3m = 4(8 + a) - (20 + 4a) = 12$ . Hence,  $4 + b = 16$ .
24. (a) We base our solution on the figure on the left below. Using that the numbers in the column with first entry 7 have sum 65, we deduce the entry  $17 - x$  in that column. Using that the numbers in the row beginning with 16 have sum 65, we deduce the entry  $x - 1$  in that row. Continuing, we obtain the entries  $16 - x$  and  $x - 2$  as well (and in that order). Since the numbers in the column with first entry 18 have sum 65, we now deduce that  $x + y = 15$ . Note that the figure on the right below shows how the square can be completed with the numbers in each row and each column summing to 65. This particular square has the numbers along its diagonals also summing to 65.

	7	25	18	
	21	19	$y$	
22	20	$16-x$	$x-2$	9
16	$17-x$	$x-1$	10	23
	$x$	6	24	

4	7	25	18	11
8	21	19	12	5
22	20	13	1	9
16	14	2	10	23
15	3	6	24	17

25. (a) The  $y$ -coordinate of both  $A$  and  $B$  is 3, and the  $y$ -coordinate of both  $C$  and  $D$  is 7. The  $y$ -coordinate of  $E$  is 5 midway between 3 and 7. Therefore, as the line segment  $\overline{GF}$  through  $E$  turns (with  $G$  remaining on  $\overline{CD}$  and  $F$  remaining on  $\overline{AB}$ ) the area swept by the segment above  $E$  and the area swept by the segment below  $E$  remain the same. Thus, the area of the trapezoid  $AFGD$  does not depend on the position of  $G$  on  $\overline{CD}$  and  $F$  on  $\overline{AB}$  (provided  $E$  is on  $\overline{GF}$ ). Taking  $G$  and  $F$  to have the same  $x$ -coordinate as  $E$ , namely  $G = (4, 7)$  and  $F = (4, 3)$ , we see that  $AF = 5$ ,  $GF = 4$ , and  $DG = 1$ . Hence, the area of trapezoid  $AFGD$  is  $(1/2) \cdot (5 + 1) \cdot 4 = 12$ .
26. (d) Let  $f(x) = 4x^{100} - 2x^2 + 3x - 1 = 4x^{100} - (x - 1)(2x - 1)$ . Since  $0.15 \leq \alpha \leq 0.65$ , we know that  $\alpha^{100}$  is very small so that  $f(\alpha) = 0$  implies that  $(\alpha - 1)(2\alpha - 1) \approx 0$ . The condition  $0.15 \leq \alpha \leq 0.65$  now strongly suggests that  $\alpha \approx 1/2$ . With this in mind, let  $\alpha = 0.5 + t$ . Then  $0.15 \leq \alpha \leq 0.65$  implies  $|t| \leq 0.35$ . Let  $w = 4\alpha^{100}$ , and observe that  $0 \leq w \leq 4(0.65)^{100}$ . Then

$$|w| = |0 - w| = |f(\alpha) - w| = |(\alpha - 1)(2\alpha - 1)| = |-0.5 + t| \cdot |2t| \geq 0.3|t|.$$

Hence,  $|t| \leq |w|/0.3 \leq 4(0.65)^{100}/0.3$ . Now, one can convince oneself that  $|t|$  must be small. For example, using  $0.65^2 < 0.7^2 < 1/2$ ,  $1/0.3 < 4$ , and  $2^{10} > 10^3$ , one obtains  $|t| < 2^2(1/2^{50})2^2 = 1/2^{46} < 1/10^{12}$ . This rather weak estimate is therefore sufficient enough to show  $\alpha$  is within  $10^{-12}$  of  $1/2$  (and it is actually within  $10^{-29}$  of  $1/2$ ).

27. (d) If  $2n + 3$  divides  $5n + 26$ , then it also divides  $2(5n + 26) - 5(2n + 3) = 37$ . Hence,  $2n + 3 \in \{\pm 1, \pm 37\}$ . Observe that the equation  $2(5n + 26) - 5(2n + 3) = 37$  implies also that if  $2n + 3 \in \{\pm 1, \pm 37\}$ , then  $2n + 3$  divides

$5n + 26$ . Since  $2n + 3 \in \{\pm 1, \pm 37\}$  corresponds to  $n$  being one of  $-20, -2, -1$ , and  $17$ , there are 4 integers  $n$  for which  $(5n + 26)/(2n + 3)$  is an integer.

28. (c) Since  $\overline{AC}$  is a diameter of the circle,  $\triangle ABC$  is a right triangle. It follows that  $AC = \sqrt{4^2 + 2^2} = 2\sqrt{5}$ . Since  $\overline{BD}$  bisects  $\angle ABC$  which is a right angle, we deduce that  $\angle CBD$  has measure  $\pi/4$  (radians). For simplicity, we write  $\angle CBD = \pi/4$  (using  $\angle$  then to also refer to measure). Also, the arc cut by the angle  $\angle CBD$  and the arc cut by the angle  $\angle CAD$  are the same so that  $\angle CAD = \pi/4$ . Let  $\theta = \angle BAC$ , and note that  $\sin \theta = BC/AC = 1/\sqrt{5}$  and  $\cos \theta = AB/AC = 2/\sqrt{5}$ . A version of the Law of Sines asserts that  $BD/\sin(\angle BAD)$  is the diameter of the circle circumscribing  $\triangle BAD$  (which is  $AC = 2\sqrt{5}$ ). Thus,

$$BD = 2\sqrt{5} \sin(\angle BAD) = 2\sqrt{5} \sin(\theta + (\pi/4)) = 2\sqrt{5} (\sin \theta \cos(\pi/4) + \cos \theta \sin(\pi/4)).$$

The latter is simplified to  $3\sqrt{2}$ .

29. (b) We want to know when

$$a + \sqrt{b} + \frac{1}{a + \sqrt{b}} = a + \frac{b + 1 + a\sqrt{b}}{a + \sqrt{b}} = 2a + \frac{b + 1 - a^2}{a + \sqrt{b}}$$

is an integer. Clearly, if  $a^2 = b + 1$ , then the last expression above is an integer. Suppose the last fraction above is an integer but  $a^2 \neq b + 1$ . Setting the fraction equal to the integer  $k$  and solving for  $\sqrt{b}$ , we obtain that  $\sqrt{b} = (b + 1 - a^2 - ka)/k$  (and note that  $k \neq 0$ ). Thus,  $\sqrt{b}$  is a rational number. Since  $b$  is a positive integer, this can only happen if  $b$  is the square of an integer so that  $\sqrt{b}$  is itself a positive integer. This means that  $a + \sqrt{b}$  is an integer and, in fact, at least 2. Hence,  $a + \sqrt{b}$  will be an integer but its reciprocal will not be so that their sum will not be. In other words,  $a + \sqrt{b}$  and its reciprocal sum to an integer precisely when  $a^2 = b + 1$ . The conditions  $1 \leq a \leq 100$  and  $1 \leq b \leq 100$  together with  $a^2 = b + 1$  imply  $2 \leq a \leq 10$  and, for each such  $a$ , there is a unique  $b$  such that the sum of  $a + \sqrt{b}$  and its reciprocal is an integer. The answer is therefore 9.

30. (e) In the drawing at the right, the large triangle is an equilateral triangle with common side length  $x$  so that its height is  $y + z = (\sqrt{3}/2)x$ . By the Pythagorean theorem,  $z = \sqrt{49 - 1} = \sqrt{48} = 4\sqrt{3}$  and  $\left(\frac{x}{2} - 1\right)^2 + y^2 = 49$ . Since  $y = (\sqrt{3}/2)x - z = \sqrt{3}\left(\frac{x}{2} - 4\right)$ , we obtain

$$49 = \left(\frac{x}{2} - 1\right)^2 + 3\left(\frac{x}{2} - 4\right)^2 = x^2 - 13x + 49.$$

In other words,  $x^2 - 13x = x(x - 13) = 0$ . Hence,  $x = 13$ .

