SOLUTIONS TO USC'S 2001 HIGH SCHOOL MATH CONTEST

1. **(d)** Since $6^{-2} = 1/36$ and $8^{1/3} = 2$, we obtain
$$\frac{5}{6^{-2} \cdot 8^{1/3}} = \frac{5 \cdot 36}{2} = 5 \cdot 18 = 90.$$

2. **(b)** Each side of the square has length $p/4$ so that $A = (p/4)^2 = p^2/16$. Since $A = 2p$, we obtain $p^2/16 = 2p$ so that $p = 32$.

3. **(d)** Using that $\sin^2\theta + \cos^2\theta = 1$ and $\sin(2\theta) = 2(\sin\theta)(\cos\theta)$, we obtain
$$(\sin\theta + \cos\theta)^2 - \sin(2\theta) = \sin^2\theta + 2(\sin\theta)(\cos\theta) + \cos^2\theta - \sin(2\theta) = 1.$$

4. **(b)** The resulting triangle will be similar to the original triangle but with dimensions increased by a factor of 1.2. If the base of the original triangle has length b and if the height is h, then the new triangle will have area
$$\frac{1}{2}(1.2 \times b)(1.2 \times h) = 1.44 \times \frac{1}{2}bh.$$
Thus, the new triangle has area that is 44% larger than the original triangle.

5. **(d)** Since $x^2 + 8x + k = (x+4)^2 + k - 16$, the equation of the parabola can be written as $y = (x+4)^2 + k - 16$. The vertex is $(-4, k-16)$ which is on the x-axis when $k = 16$.

6. **(b)** The distance in feet between the center of the pond and the center of the lake is
$$\sqrt{600^2 + 800^2} = \sqrt{200^2(3^2 + 4^2)} = 200\sqrt{25} = 1000.$$
Given the radii of the pond and lake, we deduce that the shortest path the duck can take between the two is $1000 - 30 - 700 = 270$.

7. **(e)** Since $9^{-x} = 7$, we obtain $3^{2x} = 9^x = 1/7$. Cubing, we deduce that $27^{2x} = 1/7^3 = 1/343$. It follows that $27^{2x+1} = 27/343$.

8. **(e)** There are $29 - 3 = 26$ people in the room each one of whom speaks at least one of French or English. Since 11 speak French and 24 speak English, there are exactly $11 + 24 - 26 = 9$ that speak both languages.

9. **(a)** From the given information,
$$3 = (x^2 - y^2)(x^2 - 2xy + y^2) = (x-y)(x+y)(x-y)^2 = x + y.$$
Hence, $2x = (x+y) + (x-y) = 3 + 1 = 4$ and $2y = (x+y) - (x-y) = 3 - 1 = 2$. It follows that $x = 2$ and $y = 1$ so that $xy = 2$.

10. **(b)** If one views each of the non-negative integers < 1000 as a 3-digit number (so $0 = 000$, $1 = 001$, $2 = 002$, and so on), then each digit occurs an equal number of times. Since the number of resulting digits in the 3-digit numbers is $3 \times 1000 = 3000$, each digit occurs 300 times. Alternatively, one can do a direct computation to get the answer.

11. **(c)** Observe that $\sqrt{101} - 10 = 1/(\sqrt{101} + 10)$ and
$$20 < \sqrt{101} + 10 = 20 + (\sqrt{101} - 10) = 20 + \frac{1}{\sqrt{101} + 10} < 20 + \frac{1}{20}.$$
This is sufficient to imply the answer.

12. **(c)** The formula determining x implies $x > 0$. We use that $\log_x(16x) = \log_x 4^2 + \log_x x = 2\log_x 4 + 1$ and $\log_4 \sqrt{x^{4/3}} = \log_4 x^{2/3} = (2/3)\log_4 x$. The formula then can be written as $(2/3)\log_4 x + 6\log_x 4 = 4$. We divide both sides of the equation by 2 and multiply both sides by 3. We also change the base of the logarithm; more specifically, we use that $\log_x 4 = \log_4 4/\log_4 x = 1/\log_4 x$. The formula simplifies to $(\log_4 x)^2 - 6\log_4 x + 9 = 0$. In other words, $(\log_4 x - 3)^2 = 0$. Thus, $\log_4 x = 3$ and $x = 4^3 = 64$.

13. **(d)** Of the five statements listed, the first, third, fourth, and fifth are true and the second is false. We'll let you justify this.

14. **(c)** Wilson's Theorem implies 13 divides $12! + 1$ and 7 divides $6! + 1$. Since $12! \cdot 6! + 12! + 6! + 1 = (12! + 1)(6! + 1)$, we obtain $12! \cdot 6! + 12! + 6! + 1$ is divisible by $13 \cdot 7 = 91$. Note that each of $12! \cdot 6!$, $12!$, and $6!$ is divisible by both 3 and 5. This implies $12! \cdot 6! + 12! + 6! + 1$ is not divisible by 3 and is not divisible by 5. Also, since $12! \cdot 6! + 12!$ is divisible by 11 and $6! + 1 = 721$ is not, we deduce that $12! \cdot 6! + 12! + 6! + 1$ is not divisible by 11. It follows that a correct answer is not among the choices other then (c).

15. **(a)** If the product $(x^2 + 1)(x^2 + ax + 1)(x^2 + bx + 1)$ is expanded, the coefficient of x^2 is $ab + 3$ (one can see this without expanding the product completely). Since the coefficient of x^2 in $x^6 + 1$ is 0, we obtain $ab + 3 = 0$. Hence, $ab = -3$.

16. **(d)** Since $m^2 + m - 90 = (m - 9)(m + 10)$, we have that 17 divides $m^2 + m - 90$ if and only if 17 divides $m - 9$ or $m + 10$. The m with $10 \le m \le 100$ such that $m - 9$ is divisible by 17 are 26, 43, 60, 77, and 94. The m with $10 \le m \le 100$ such that $m + 10$ is divisible by 17 are 24, 41, 58, 75, and 92. Hence, the answer is 10.

17. **(e)** Suppose the dice are red, green, and blue (just to refer to them). Initially, suppose also that red is the "remaining die" in the problem. For each $k \in \{1, 2, 3, 4, 5, 6\}$, the probability that k is rolled by the red die is $1/6$. The probability that the other two dice show numbers summing to k is $(k - 1)/36$ (the pair (g, b) where g is the number rolled by the green die and b is the number rolled by the blue die must be one of the $k - 1$ pairs $(1, k - 1), (2, k - 2), \dots, (k - 1, 1)$). Thus, the probability that the green die and blue die show numbers summing to the number shown on the red die is

$$\sum_{k=1}^{6} \frac{1}{6} \times \frac{k-1}{36} = \frac{5}{72}.$$

A similar analysis holds if the blue or the green die is the remaining die. The answer is therefore $3 \times 5/72 = 5/24$.

18. **(d)** Observe that $x^4 + 6x^2 + 9 = (x^2 + 3)^2$ and $36x^2 - 72x + 36 = 36(x - 1)^2 = (6x - 6)^2$. Rewriting the equation in x, we obtain $(x^2 + 3)^2 - (6x - 6)^2 = 0$. Since $a^2 - b^2 = (a + b)(a - b)$, we deduce $(x^2 + 3)^2 - (6x - 6)^2 = (x^2 + 6x - 3)(x^2 - 6x + 9) = 0$. Note that $x^2 - 6x + 9 = (x - 3)^2$ is a polynomial with one real root. The quadratic formula implies $x^2 + 6x - 3$ has two real roots neither of which is 3. Hence, the original equation has 3 distinct real solutions.

19. **(b)** At 2:00, the lady is $100X + Z$ miles from home. In 18 minutes she travels $100X + Z - (10Z + X) = 99X - 9Z$ miles. So she has traveled at a rate of $11X - Z$ miles every 2 minutes. In 60 minutes she travels $100X + Z - (10X + Z) = 90X$ miles. So she has traveled at a rate of $3X$ miles every 2 minutes. Thus, $11X - Z = 3X$ so that $Z = 8X$. Since X and Z are digits with $X \ge 1$, we get $X = 1$ and $Z = 8$. Hence, at 2:00 she was 108 miles from home and she traveled at a constant speed of 3 miles each 2 minutes. It follows that it took her $36 \times 2 = 72$ minutes to get home. This implies the answer.

20. **(b)** At the first time that the hands are perpendicular after 5:00, let θ in radians be the smallest angle that the minute hand makes with a line ℓ passing through the center of the hand's position at noon. Then the smallest angle between ℓ and the hour hand is $2\pi(5/12) + (\theta/12)$. It follows that $(5\pi/6) + (\theta/12) - \theta = \pi/2$ so that $\pi/3 = 11\theta/12$. Hence, $\theta = 4\pi/11$. The number of hours elapsed is $(4\pi/11)/(2\pi) = 2/11$.

21. **(c)** After the first sock is chosen, there are 63 socks left, 7 of which are the same color as the first sock. Thus, the probability is $7/63 = 1/9$.

22. **(a)** As $1-2=-1$, $3-4=-1$, $5-6=-1$, and so on, we see that $1-2+3-4+5-6+\cdots+(n-2)-(n-1) = -(n-1)/2$. Hence, the sum in the problem is $n-(n-1)/2=(n+1)/2$. If this is equal to 2001, we obtain $n=4001$. The answer is $4+0+0+1=5$.

23. **(d)** Since AB is prime, B is either 1 or 3. Also, $A+C \le 11$, so $E=1$. It follows that $B=3$. Since $D \le 6$, $B+D<10$ and $A+C=10$. We deduce that A and C must be 4 and 6 in some order. Since 63 is not prime, $AB=43$. The answer is $4+3=7$.

24. **(b)** We use that $(2y)^2=(2x)(2z)$ and $AD=BC$. Since $2y$ is twice the area for ΔBCE, we deduce $2y=EC\times BC$. Similarly, $2x=DE\times AD=DE\times BC$ and $2z=(DE+EC)\times BC$. Hence, $EC^2BC^2=DE\times BC\times(DE+EC)\times BC$ so that $EC^2=DE^2+DE\times EC$. Dividing by EC^2, we see that DE/EC is a solution of $1=x^2+x$. By the Quadratic Formula, x^2+x-1 has $(-1+\sqrt{5})/2$ as its only positive root. Hence, $DE/EC=(-1+\sqrt{5})/2$.

25. **(d)** If Statement (4) is true, then Statement (3) is false. Hence, (3) and (4) cannot both be true. It follows that Statement (1) cannot be true. Hence, both Statement (1) and Statement (3) must be false. We see then that Statement (4) and Statement (5) are true. It follows that Statement (2) is true. The answer is 3.

26. **(b)** Using $\sin A^\circ=\cos(90-A)^\circ$, we deduce that $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \cos 80^\circ \cos 40^\circ \cos 20^\circ$. Using $\sin(2\theta) = 2\sin\theta\cos\theta$ (with $\theta=20^\circ$, then $\theta=40^\circ$, and then $\theta=80^\circ$) and $\sin A^\circ=\sin(180-A)^\circ$, we obtain

$$8\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ = 4\sin 40^\circ \cos 40^\circ \cos 80^\circ = 2\sin 80^\circ \cos 80^\circ = \sin 160^\circ = \sin 20^\circ.$$

Dividing through by $8\sin 20^\circ$, we deduce that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = 1/8$. Hence, the answer is $1/8$.

27. **(d)** The area of one-fourth of the region can be seen to be the difference between the area of a sector with radius 6 and internal angle $\pi/6$ radians and the area of a right triangle with one side $3\sqrt{3}$ and hypotenuse 6. The area of the sector is $(1/2)(\pi/6)6^2=3\pi$. The area of the triangle is $(1/2)(3\sqrt{3})3=9\sqrt{3}/2$. Hence, the answer is $4\times(6\pi-9\sqrt{3})/2=12\pi-18\sqrt{3}$.

28. **(e)** We compare first the area $\mathcal{A}$ of ΔADF and the area $\mathcal{A}'$ of ΔABC, and we use that $\mathcal{A}'=1$. Let h denote the length of the altitude $\overline{FF'}$ drawn from F in the first triangle and h' the length of the altitude $\overline{CC'}$ drawn from C in the second triangle. Then $\Delta FAF'$ is similar to $\Delta CAC'$. Since $CF/FA=2/3$, we see that $FA=(3/5)CA$. Hence, $h=(3/5)h'$. Also, $AD/DB=2/3$ implies $AD=(2/5)AB$. It follows that $\mathcal{A}=(1/2)h(AD)=(6/25)(1/2)h'(AB)=(6/25)\mathcal{A}'=6/25$. Similarly, we deduce that each of ΔEDB and ΔCFE has area $6/25$. It follows that ΔDEF has area $1-3(6/25)=7/25$.

29. **(c)** The set $\{0,1,3,4\}$ shows that $n\ge 5$. Each positive integer has a remainder of 0, 1, or 2 when divided by 3. If S has 5 or more elements, then at least one of the following must be true:

    (i) Three of them, say a, b, and c, have the same remainder when divided by 3.

    (ii) Three of them, say a, b, and c, have different remainders when divided by 3.

    If (i) holds and r is the common remainder, then there are integers k, ℓ, and m such that $a=3k+r$, $b=3\ell+r$, and $c=3m+r$. In this case, $a+b+c=3(k+\ell+m+r)$ is divisible by 3. If (ii) holds, then there are integers k, ℓ, and m such that $a+b+c=3k+3\ell+3m+0+1+2=3(k+\ell+m+1)$, a number divisible by 3. Thus, if S has 5 or more elements, then there are 3 elements whose sum is divisible by 3.

30. **(a)** Let A, B, and C be the vertices of the triangle with $AB=AC$. Let D be the center of the smaller circle and E the center of the larger circle. Let F be such that $\overline{AF}$ is an altitude for ΔABC. Let $h=AF$ and $x=FC$ so that the area of ΔABC is hx. Drawing radii for the circles from their centers to the side $\overline{AC}$, we obtain two smaller right triangles each having its hypotenuse on $\overline{AF}$ and each similar to ΔAFC. Thus, $AD/1=(AD+3)/2=AC/FC=\sqrt{h^2+x^2}/x$. The first of these equations implies $AD=3$. Since $h=AD+5$, we obtain $h=8$. Hence, $3=\sqrt{64+x^2}/x$ so that $9x^2=64+x^2$. Thus, $x^2=8$ and $hx=8\times\sqrt{8}=16\sqrt{2}$.