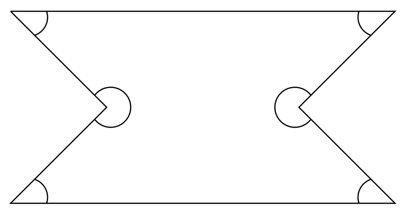
High School Math Contest
University of South Carolina
December 5, 1992

1. A loaded die shows the numbers 1, 2, 3, 4, 5, 6 with frequencies 1/8, 1/8, 1/24, 1/24, 1/24, 5/8, respectively. If such a die is rolled, what is the probability that an odd number appears?

   (a) 1/6 (b) 5/24 (c) 1/4 (d) 3/8 (e) 1/2

2. What is the sum of the interior angles in the figure below?



   (a) $180°$ (b) $360°$ (c) $450°$ (d) $540°$ (e) $720°$

3. If the product of five integers is a multiple of 32, then what is the smallest number of these integers that must be even?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) All

4. The smallest positive integer n for which the decimal expansion of $n!$ ends in 3 zeroes is

   (a) 10 (b) 12 (c) 14 (d) 15 (e) 16

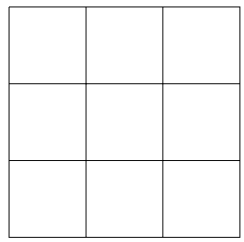
5. If $x + 3y + 5z = 200$ and $x + 4y + 7z = 225$, then $x + y + z =$

   (a) 90 (b) 125 (c) 150 (d) 175 (e) None of these

6. $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \cdots + \dfrac{1}{99 \times 100} =$ ______

   (a) 99/100 (b) 1 (c) 100/99 (d) 101/99 (e) 3/2

7. The first 9 odd numbers $1, 3, 5, \ldots, 17$ are put into the 9 squares below, one in each square, in such a way that the sum along any row or diagonal is the same. What is the common sum?



(a) 26 (b) 27 (c) 28 (d) 29 (e) 30

8. For $-\pi/2 < x < \pi/2$, let $f(x) = \log_{10}(\tan(x) + \sec(x))$. Then the function $g(x) = \dfrac{10^{f(x)} - 10^{-f(x)}}{2}$ can be more simply expressed as:

(a) $\sin(x)$ (b) $\sin(2x)$ (c) $\sin(f(x))$ (d) $\tan(x)$ (e) $\sec(x)$

9. If A, B, and C are constants such that for all values of x,

$$3x^2 + 6x + 5 = (Ax + B)(x - 3) + C(x^2 + 1),$$

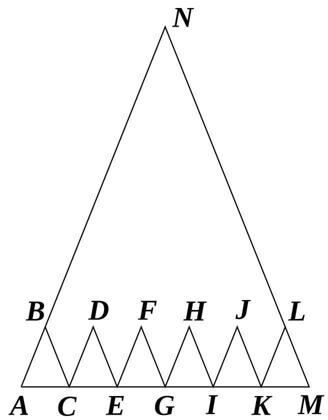
what does A have to be?

(a) 0 (b) -2 (c) 2 (d) -5 (e) 5

10. A contest among $n \geq 2$ players is held over a period of 4 days. On each day each player receives a score of $1, 2, \ldots, n$ points with no two players getting the same score on a given day. At the end of the contest it is discovered that every player received the same total of 26 points. How many players participated?

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

11. The triangles $\triangle ABC$, $\triangle CDE$, $\triangle EFG$, $\triangle GHI$, $\triangle IJK$, and $\triangle KLM$ below are congruent to one another and similar to $\triangle ANM$. What's the ratio of the area of $\triangle ANM$ to $\triangle ABC$?



(a) 18 (b) 20 (c) 24 (d) 32

(e) none of the above

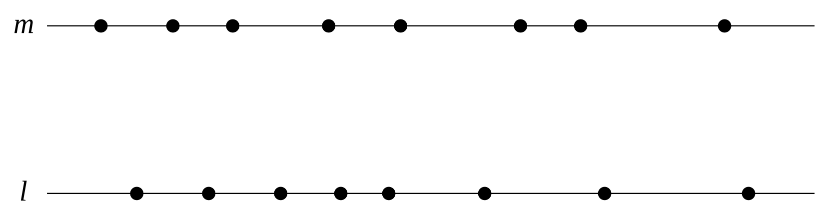
12. Let $a_1 = 1$ and $a_{n+1} = (3a_n + 4)/(2a_n + 3)$ for $n \geq 1$ (so $a_1 = 1$, $a_2 = 7/5$, $a_3 = 41/29, \ldots$). Then as n approaches infinity, the value of a_n approaches

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\dfrac{2+\sqrt{3}}{3}$ (d) $4\sqrt{3}/3$ (e) ∞

13. For $1 \leq k \leq 1000$, the number of values of k for which $3k + 2$ is a perfect square is

(a) 0 (b) 5 (c) 7 (d) 9 (e) 15

14. If each of the 8 indicated points on the line ℓ is joined by a straight line segment to each of the 8 indicated points on the line m, and if no three of the resulting line segments meet in a point between the lines ℓ and m, then how many interior points of intersection are there among these 64 segments?



(a) 784 (b) 824 (c) 962 (d) 1024 (e) 4096

15. What is the greatest common divisor of $2^{15} + 3^{15}$ and $2^{25} + 3^{25}$?

(a) 5 (b) 11 (c) 55 (d) 275 (e) a number > 300

16. Given that $1025/1024 = 1.0009765625$, what is the sum of the digits of 5^{10} ?

(a) 36 (b) 40 (c) 41 (d) 50 (e) 102

17. Points (x_k, y_k) on the graph of $y = \cos x$ are chosen so that $x_0 = 0$, $x_1 = \pi/100$, $x_2 = 2\pi/100$, $x_3 = 3\pi/100$, $\ldots$, $x_{100} = 100\pi/100 = \pi$. What is the value of $\sum_{k=1}^{100} (y_k - y_{k-1})$?

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

18. A group of people are playing cards with an incomplete deck of cards. When 7 cards are dealt to each person, there are 10 cards left over. When 8 cards are dealt to each person, there are 5 cards left over. If x cards are dealt to each person and no cards are left over, then what is x ?

(a) 4 (b) 5 (c) 9 (d) 10 (e) 11

19. Suppose that $*$ is an associative operation on a set S. Define x^n to mean $x*x*x*\cdots*x$, n times. (So, for example, $x^3 = x*x*x$.) Suppose further that an element a of S is such that all of $a, a^2, \ldots, a^9$ are different but $a^{10} = a^3$. Then there is some element b belonging to S such that $b = b^2$. One such element b is:

(a) a^4 (b) a^5 (c) a^7 (d) a^9 (e) a^{13}

20. Let x be the smallest number larger than 2 that leaves a remainder of 2 when divided by each of 3, 5, and 6. Then the sum of the digits of x is

(a) 1 (b) 5 (c) 11 (d) 12 (e) 14

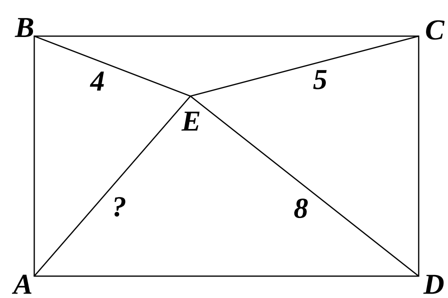
21. The decimal expansion of $(1/2)^{100}$ has for its third nonzero digit from the right the digit

(a) 1 (b) 3 (c) 4 (d) 6 (e) 9

22. Given that $\log_{10} 2 = 0.3010299957\ldots$ and $\log_{10} 3 = 0.4771212547\ldots$, what is the number of digits in the decimal expansion of 12^{10} ?

(a) 9 (b) 10 (c) 11 (d) 12 (e) 13

23. Find the length of the line segment AE in the rectangle below.



(a) $\sqrt{60}$ (b) 6.4 (c) 7 (d) $\sqrt{55}$ (e) 7.4

24. Let $a_1 = \sqrt{2}$. For $n \geq 2$, define $a_n = \sqrt{2 + a_{n-1}}$. Then $a_4 = 2\cos\theta$ for which of the following values of θ ?

(a) $\pi/12$ (b) $\pi/16$ (c) $\pi/18$ (d) $\pi/24$ (e) $\pi/32$

25. Suppose a cylindrical open-topped can has radius 3 cm and height 10 cm. If the can is tilted at a 45° angle, what volume of water (in cm^3) will it hold?

(a) 45π (b) 63π (c) 70π (d) 75π (e) 80π

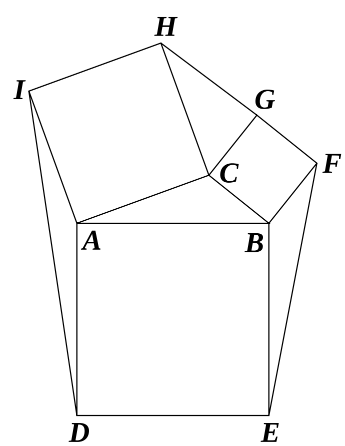
26. Simplified, $(1 \cdot 3 \cdot 5 \cdots 31) \cdot 2^{16} \cdot 16!$ becomes

(a) 30! (b) 31! (c) 32! (d) $(30!)^2$ (e) 36!

27. Which of the following has the largest value?

(a) $\sin(\sin 20°)$ (b) $\sin(\cos 20°)$ (c) $\cos(\sin 20°)$
(d) $\cos(\cos 20°)$ (e) $\cos 20°$

28. In the drawing below, $\angle CAB = 20°$, $\angle ABC = 40°$, $\angle BCA = 120°$, and the three quadrilaterals $BADE$, $CBFG$, and $ACHI$ are squares. Determine which of the triangles $\triangle ABC$, $\triangle BEF$, $\triangle CGH$, and $\triangle AID$ has a larger area than the rest.

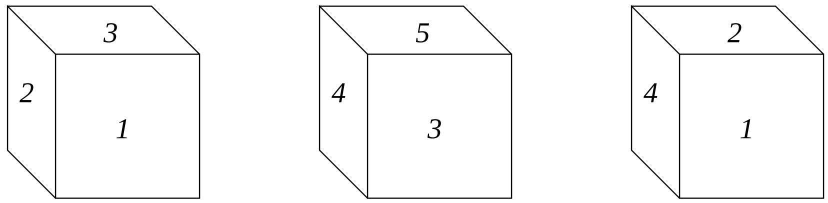


(a) $\triangle ABC$ (b) $\triangle BEF$ (c) $\triangle CGH$
(d) $\triangle AID$ (e) All have the same area

29. There is a mathematical theorem that states that the binonial coefficient $\binom{n}{m}$ is odd if and only if the binary expansion of n has a 1 in every position in which m has a 1. How many of the binomial coefficients $\binom{54}{0}$, $\binom{54}{1}$, $\binom{54}{2}$, $\ldots$, $\binom{54}{54}$ are odd?

(a) 5 (b) 8 (c) 12 (d) 16 (e) 27

30. The figure below shows three views of the same numbered cube. One number actually occurs twice on the cube. Also, the number that appears twice is not on the bottom of any of the views. What number appears twice?



(a) 1 (b) 2 (c) 3 (d) 4 (e) 5